Double Roman Domination on graphs with maximum degree 3

11th Latin American Workshop on Cliques in Graphs

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Campus de Quixadá @ Universidade Federal do Ceará

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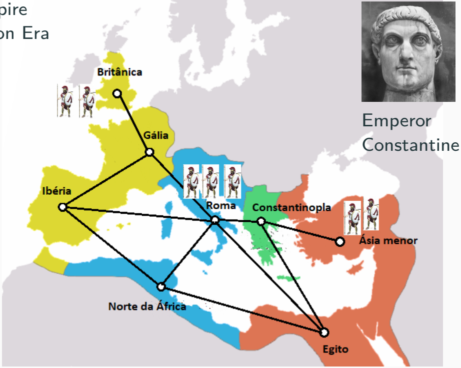
Double Roman Domination

Double Roman Domination (Beeler et al. 2016 [2])

Roman Empire
330 Common Era



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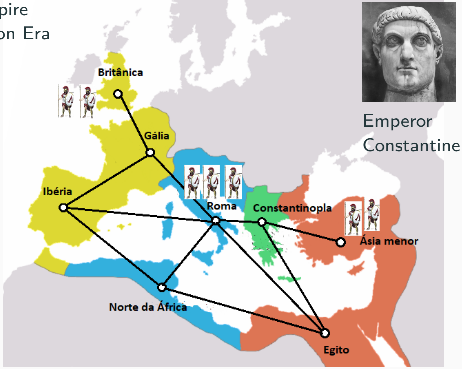

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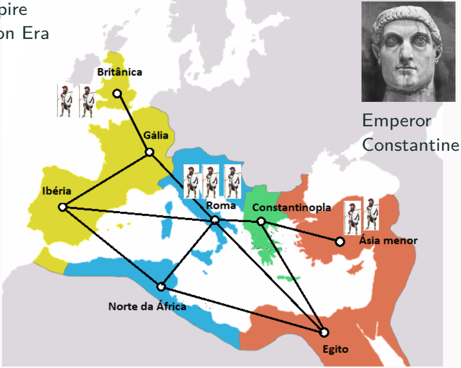

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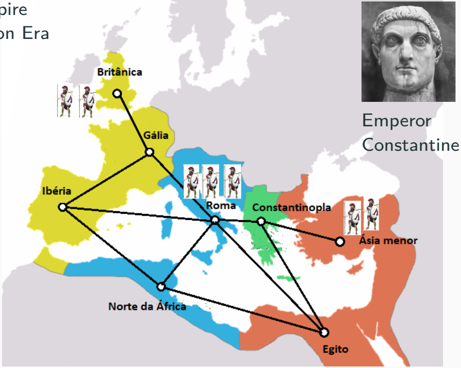

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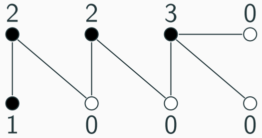
- If a region has fewer than 2 troops, then it must have “stronger” neighbors.
- Troops can move to a neighboring region only if at least one troop remains at the origin.

Double Roman Dominating Function

Given a simple graph G, a function $f\colon V(G) \to \{0,1,2,3\}$ is a **Double Roman Dominating Function** (DRDF) of G if:



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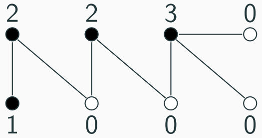

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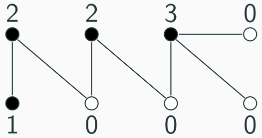

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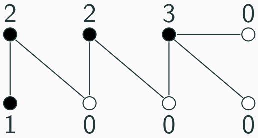

A DRDF partitions $V(G)$ into at most 4 subsets V_0, V_1, V_2, V_3.

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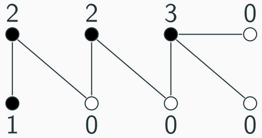

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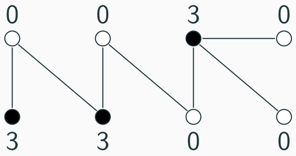
- Alternative notation for DRDF: $f = (V_0, V_1, V_2, V_3)$

Double Roman domination number — $\gamma_{dR}(G)$

The **weight** of a DRDF f is the sum of the labels of the vertices of G under f and is denoted by $\omega(f)$.



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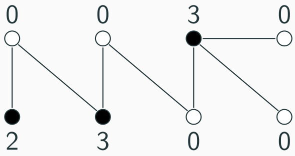

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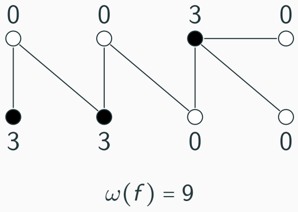

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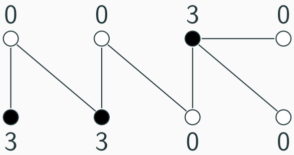

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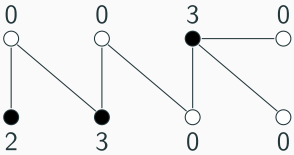

$\omega(f') = 8$

- The **double Roman domination number** is the least weight of a DRDF of G and is denoted by $\gamma_{dR}(G)$.
- Given $k \in \mathbb{N}$, deciding whether an arbitrary G has $\gamma_{dR}(G) \leq k$ is NP-Complete, even when restricted to bipartite, chordal and planar graphs.

Graphs with maximum degree 3

- Ahangar et al. [1] completely determined $\gamma_{dR}(G)$ for graphs with maximum degree 2 (paths and cycles).

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- Question 1: Are there families of cubic graphs for which $\gamma_{dR}(G) < n$?
- Question 2: Is determining $\gamma_{dR}(G)$ for graphs with maximum degree 3 an NP-Complete problem?

Result 1: NP-Completeness

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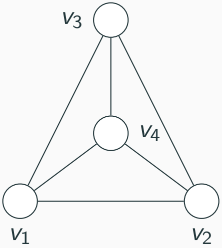

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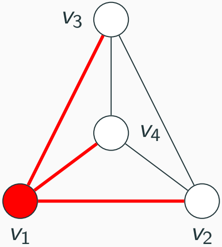

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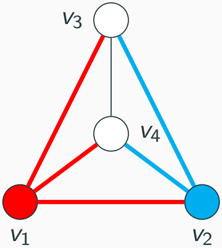

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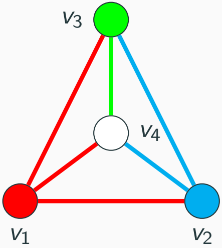

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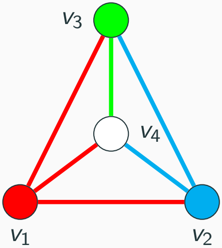

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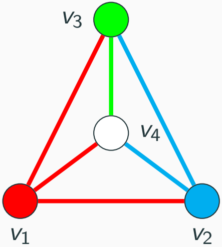

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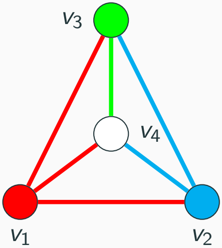

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- **VERTEX COVER PROBLEM (VCP)**: given a graph G and $\ell \in \mathbb{N}$, decide whether G has a vertex cover S with $|S| \leq \ell$.
- B. Mohar [4] showed that VCP is $\mathcal{NP}$-complete even when restricted to 2-connected planar cubic graphs.

Double Roman Domination Problem

DOUBLE ROMAN DOMINATION PROBLEM (DRDP)

Instance: A graph $G = (V, E)$ and a positive integer k.
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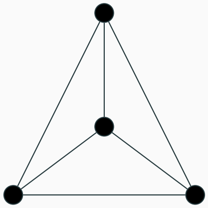
- **Step 1:** Show that DRDP is NP (easy)
- **Step 2:** Show that DRDP is NP-Hard. We show that there exists a polinomial time reduction from VCP to DRDP.

Illustration of the NP-Hardness Proof

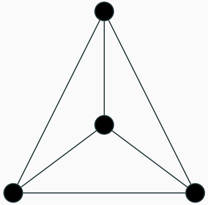
- The input to VERTEX-COVER-PROBLEM is a 2-connected planar 3-regular G.



Example of graph G

Illustration of the NP-Hardness Proof

- The input to `VERTEX-COVER-PROBLEM` is a 2-connected planar 3-regular G.
- We take G and substitute each of its edges $e = uv$ by the gadget G_{uv} below, creating a new graph H.



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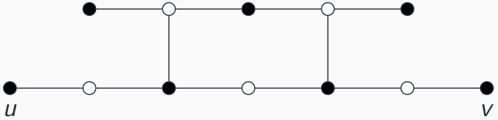
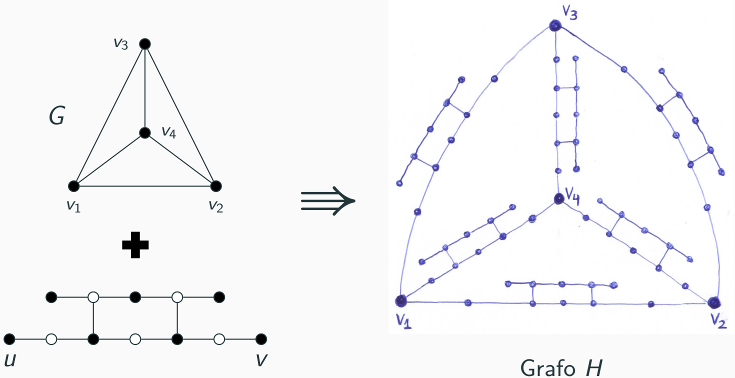


Gadget G_{uv}



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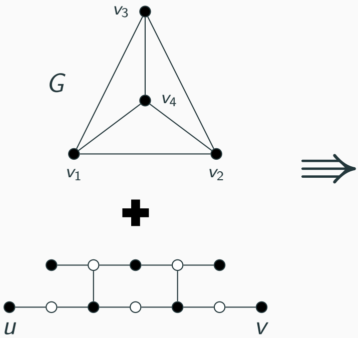

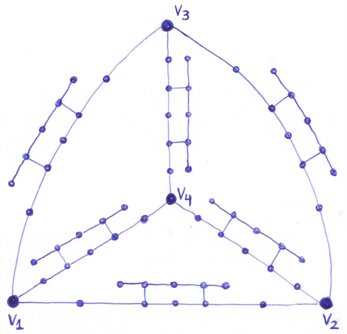

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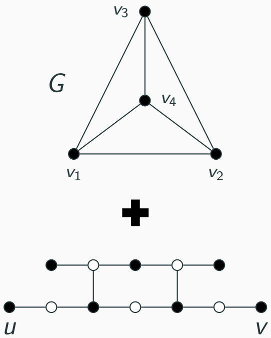
- Gadget G_{uv} is bipartite and u, v are in the same part.



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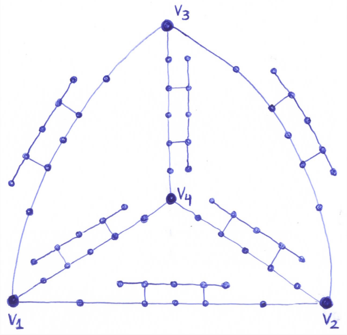

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- Thus, the resulting graph H is also bipartite.

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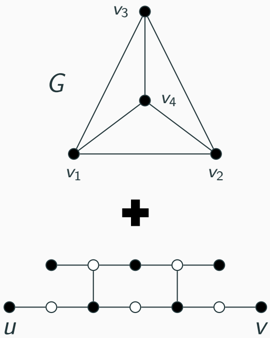

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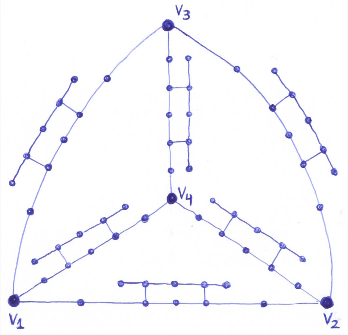

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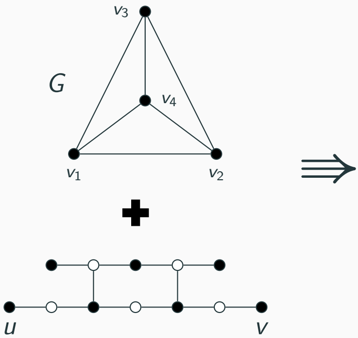

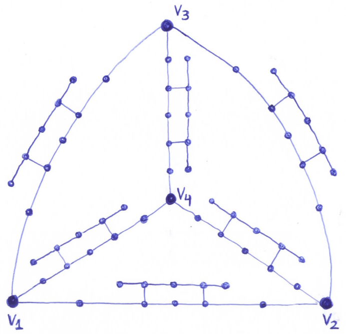

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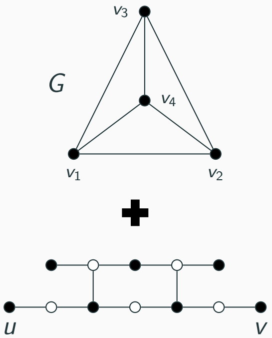

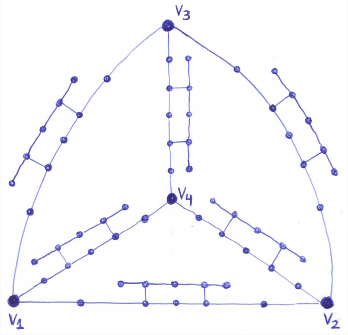

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- We prove that $\gamma_{dR}(H) = \tau(G) + 2|V(G)| + 8|E(G)|$.

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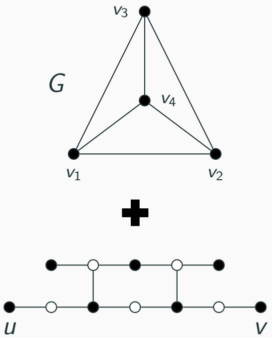

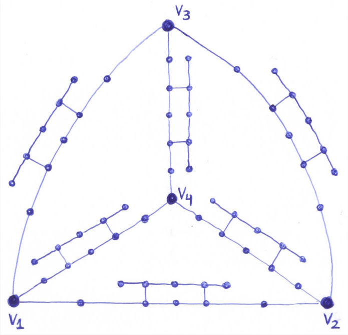

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- We prove that $\gamma_{dR}(H) = \tau(G) + 2|V(G)| + 8|E(G)|$.
- Since $\tau(G)$ is hard, then $\gamma_{dR}(H)$ is also hard.

Illustration of the NP-Hardness Proof

Lemma 3

Given a 2-connected planar cubic graph G, let H be a graph constructed from G by replacing each edge uv in G by a gadget G_{uv}. Then, $\gamma_{dR}(H) = \tau(G) + 2|V(G)| + 8|E(G)|$.

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- **Lemma [2]**: In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the label 1.
- We construct a DRDF $f\colon V(H) \to \{0, 2, 3\}$ to H with weight $\omega(f) = \tau(G) + 2|V(G)| + 8|E(G)|$.

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- For each gadget G_{uv} of H, either $u \in C$, or $v \in C$, or both.

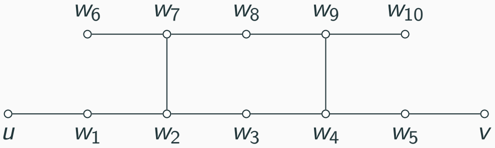

Illustration of the NP-Hardness Proof

Lemma 3

Given a 2-connected planar cubic graph G, let H be a graph constructed from G by replacing each edge uv in G by a gadget G_{uv}. Then, $\gamma_{dR}(H) = \tau(G) + 2|V(G)| + 8|E(G)|$.

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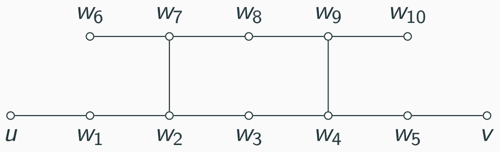

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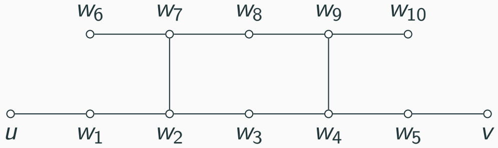

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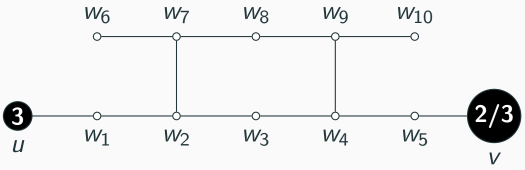

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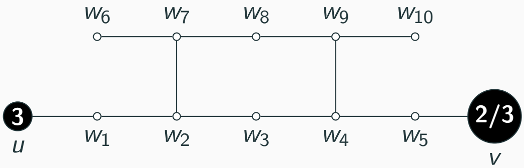
Outline of the proof of the upper bound:

- Assign $f(w_7) = 3$, $f(w_4) = 3$ and $f(w_{10}) = 2$.



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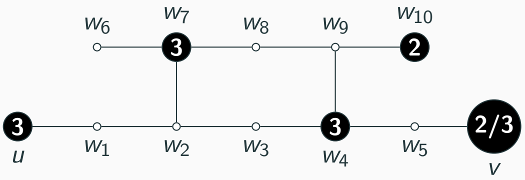

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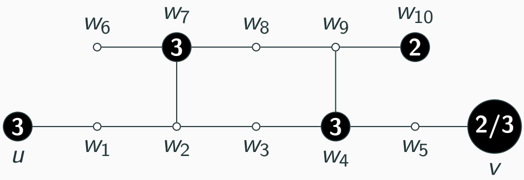
- Assign $f(w_7) = 3$, $f(w_4) = 3$ and $f(w_{10}) = 2$.
- Assign $f(x) = 0$ for every remaining unlabeled vertex $x \in G_{uv}$.



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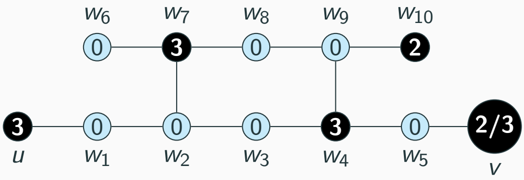
- Assign $f(x) = 0$ for every remaining unlabeled vertex $x \in G_{uv}$.
- Now, we are ready to count the weight of f



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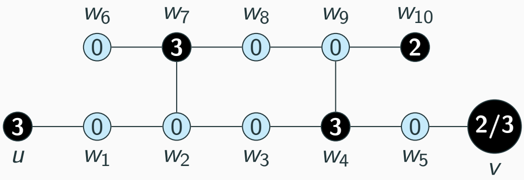
Outline of the proof of the upper bound:

$$\gamma_{dR}(H) \leq \omega(f) = 3|C| + 2(|V(G)| - |C|) + 8|E(G)|$$



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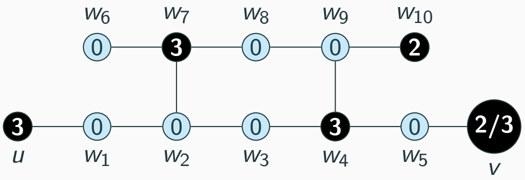

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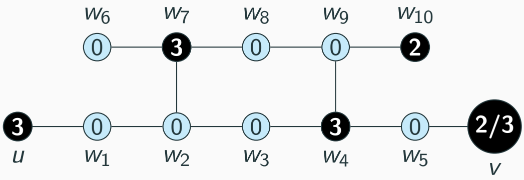
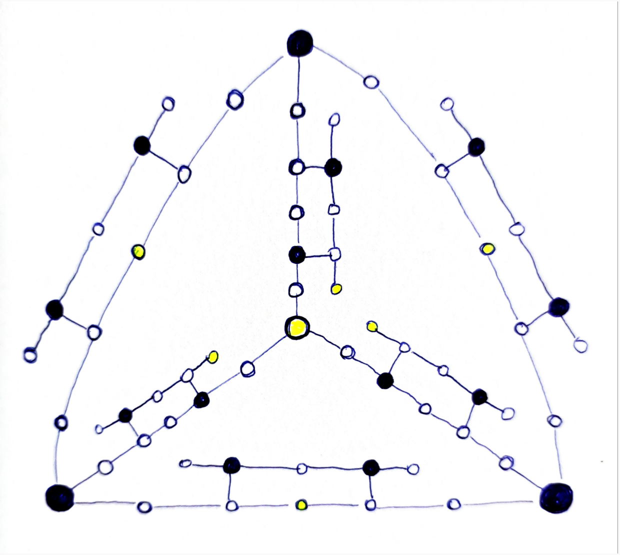
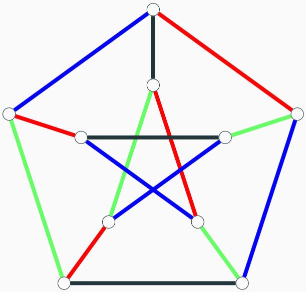


Illustration — Final DRDF of H



Result 2: Double Roman Domination on Snarks

Snarks are connected 3-regular graphs without cut edges and that do not admit a proper edge coloring with 3 colors.



Petersen Graph

Snarks

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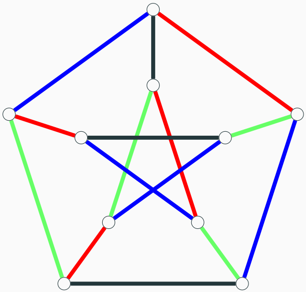

Petersen Graph

- Motivation: Four-Color Conjecture

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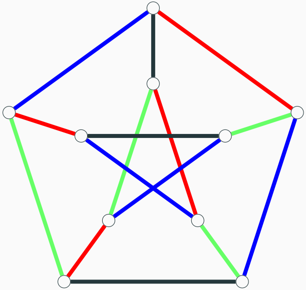

Petersen Graph

- Motivation: Four-Color Conjecture
- 2020 – Pereira [5]
  - $\gamma(G)$: flower snarks, Goldberg snarks, Blanuša snarks, Loupekine snarks

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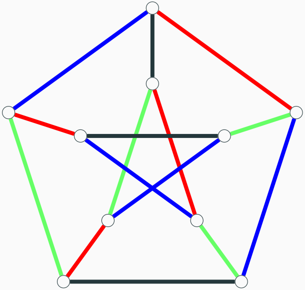

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- Motivation: Four-Color Conjecture
- 2020 – Pereira [5]
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- 2022 – Luiz and da Hora [3]
  - $\gamma_R(G)$: Goldberg snarks, Blanuša snarks, Loupekine snarks

Generalized Blanuša Snarks

- Two infinity families of snarks constructed by Watkins [8] in 1983.
  - $\mathfrak{B}^1 = \{B_1^1, B_2^1, B_3^1, \dots\}$
  - $\mathfrak{B}^2 = \{B_1^2, B_2^2, B_3^2, \dots\}$

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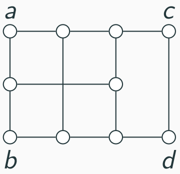
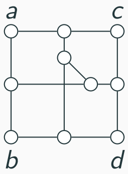
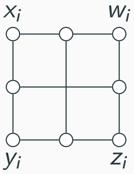
  - $\mathfrak{B}^1 = \{B_1^1, B_2^1, B_3^1, \dots\}$
  - $\mathfrak{B}^2 = \{B_1^2, B_2^2, B_3^2, \dots\}$
- They are constructed in a recursive way by using subgraphs called **construction blocks**:



Block A_1



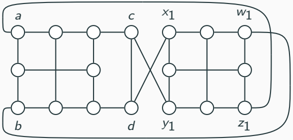
Block A_2



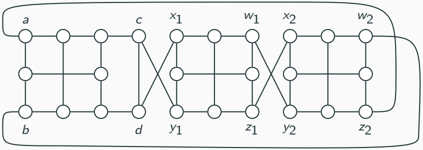
Linkage block L_i

Base cases of the recursive construction

The first two smallest snarks of the family $\mathfrak{B}^1$:

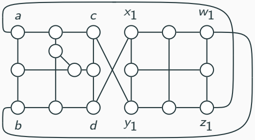


Snark B_1^1

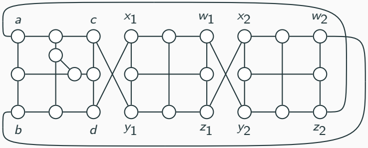


Snark B_2^1

The first two smallest snarks of the family $\mathfrak{B}^2$:



Snark B_1^2

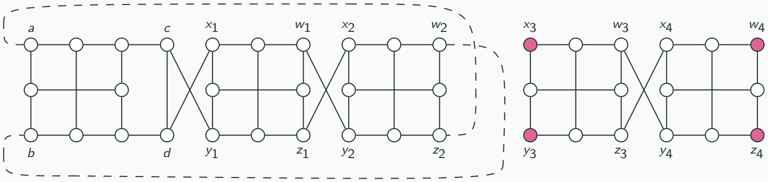


Snark B_2^2.

Inductive Step of the recursive construction

Example: First family of Generalized Blanuša Snarks

- We want to construct Snark B_4^1.
- B_4^1 is constructed from snark B_2^1 and the linkage block L_4.



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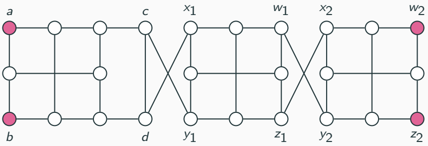
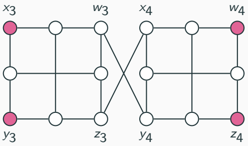
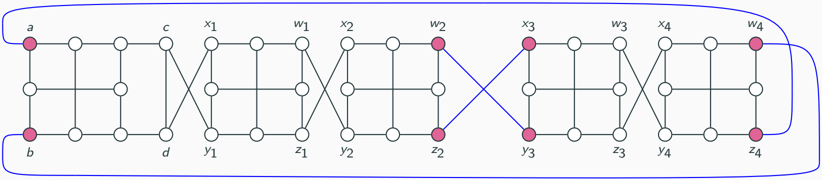




Inductive Step of the recursive construction

Example: First family of Generalized Blanuša Snarks

- We want to construct Snark B^1_4.
- B^1_4 is constructed from snark B^1_2 and the linkage block L_4.



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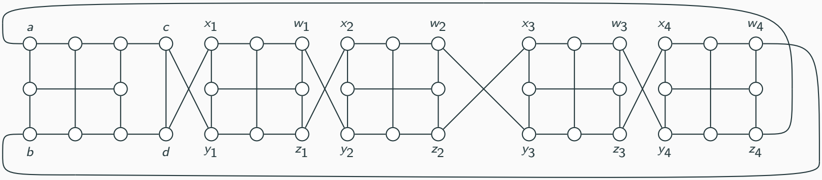

Result

Theorem 4

If B_i^k is a Generalized Blanuša Snark with $k \in \{1,2\}$ and $i \geq 1$, then

$$\gamma_{dR}(B_i^k) = \begin{cases} 6i + 10 & \text{if } k = 1, i \geq 3 \text{ and } i \text{ odd}; \\ 6i + 9 & \text{otherwise.} \end{cases}$$

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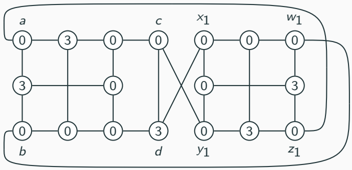
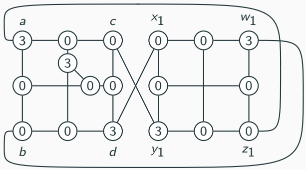
- The proof of the lower bound is a proof by contradiction.
- Due to time constraints, we only show the upper bound, which is an inductive proof.

Illustration of the proof — Special Cases



Snark B_1^1 with a DRDF f with weight $\omega(f) = 6 \cdot 1 + 9 = 15$.



Snark B_1^2 with a DRDF f with weight $\omega(f) = 6 \cdot 1 + 9 = 15$.

Illustration of the proof — Base Case of Inductive Proof

A DRDF f_i for a generalized Blanuša snark B_i^k is called **special** if $f_i(a) = 3,\ f_i(b) = 0,\ f_i(w_i) = 3,\ f_i(z_i) = 0$ and has weight

$$\omega(f_i) = \begin{cases} 6i + 10 & \text{if } k = 1, i \geq 3 \text{ and } i \text{ odd;} \\ 6i + 9 & \text{otherwise.} \end{cases}$$

We have four base cases.

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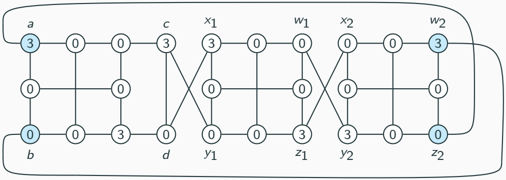

Snark B_2^1 with a special DRDF f_2 with weight $\omega(f_2) = 6 \cdot 2 + 9 = 21$.

Illustration of the proof — Base Case of Inductive Proof

A DRDF f_i for a generalized Blanuša snark B_i^k is called **special** if $f_i(a) = 3,\ f_i(b) = 0,\ f_i(w_i) = 3,\ f_i(z_i) = 0$ and has weight

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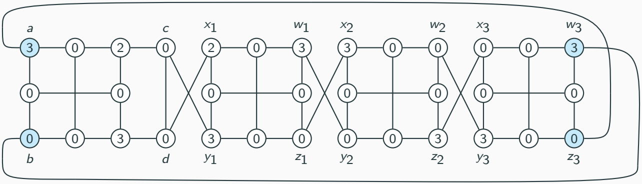
We have four base cases.

Base Case 2: Snark B_3^1.



Snark B_3^1 with a special DRDF f_3 with weight $\omega(f_3) = 6 \cdot 3 + 10 = 28$.

Illustration of the proof — Base Case of Inductive Proof

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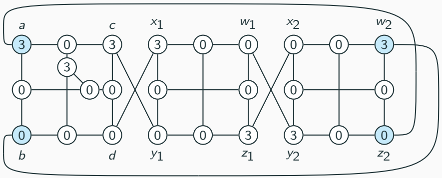
$$\omega(f_i) = \begin{cases} 6i + 10 & \text{if } k = 1, i \geq 3 \text{ and } i \text{ odd;} \\ 6i + 9 & \text{otherwise.} \end{cases}$$

We have four base cases.

Base Case 3: Snark B_2^2.



Snark B_2^2 with a special DRDF f_2 with weight $\omega(f_2) = 6 \cdot 2 + 9 = 21$.

Illustration of the proof — Base Case of Inductive Proof

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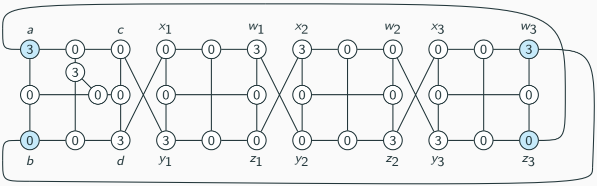
We have four base cases.

Base Case 4: Snark B_3^2.



Snark B_3^2 with a special DRDF f_2 with weight $\omega(f_2) = 6 \cdot 2 + 9 = 27$.

Illustration of the proof — Induction Step

- We illustrate the induction step for snarks B_i^k with $k = 1$.

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Illustration of the proof — Induction Step

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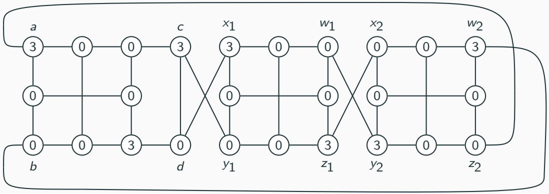
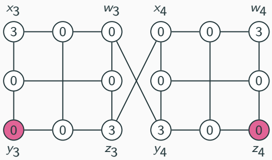
- The remaining case $k = 2$ is similar.
- **Let us construct a special DRDF for B_4^1:**



Special DRDF of B_2^1



Partial labeling of the linkage graph L_4

Illustration of the proof — Induction Step

- Remove the out-edges from B_2^1.

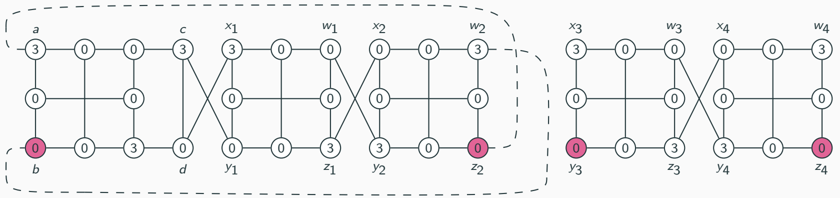


Illustration of the proof — Induction Step

- Remove the out-edges from B_2^1.
- Some vertices with label 0 do not have neighbors with label 2.

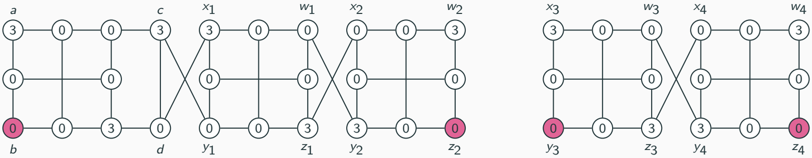


Illustration of the proof — Induction Step

- Add the input-edges linking specific pairs of degree-2 vertices in B_2^1 and L_4.

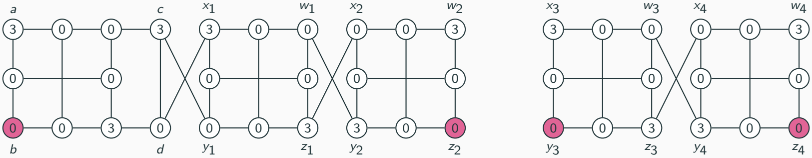


Illustration of the proof — Induction Step

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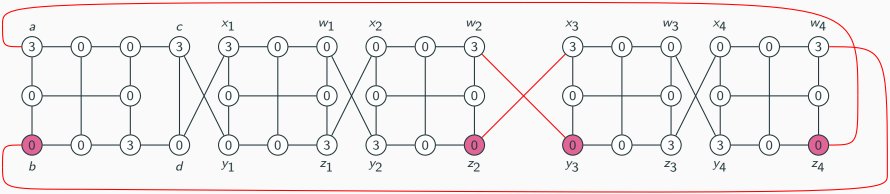
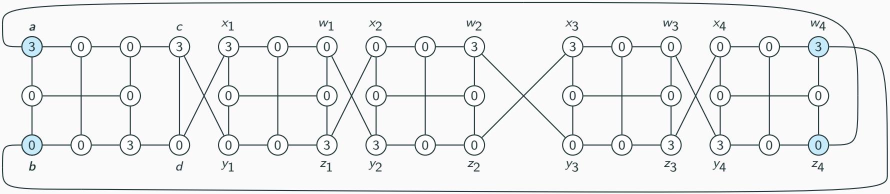


Illustration of the proof — Induction Step

- At the end we have B_4^1 with a special DRDF f_4.
  - since $f_4(a) = 3,\ f_4(b) = 0,\ f_4(w_4) = 3,\ f_4(z_4) = 0.$



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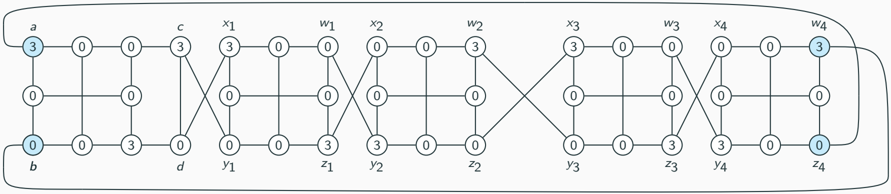

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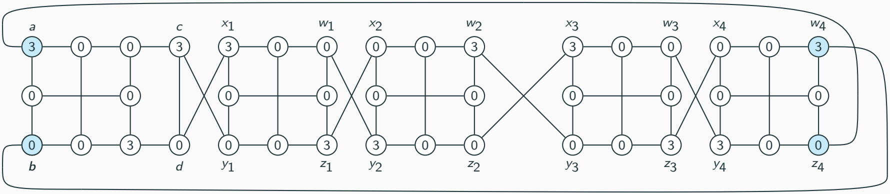
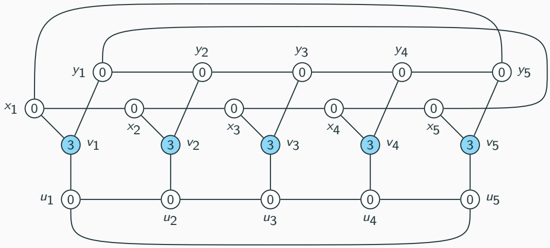
- This concludes the inductive construction. ■



Flower Snarks — Result

The infinity family of Flower Snarks comprises the graphs $F_3, F_5, \ldots, F_i$, with i odd and $i \geq 3$.



Flower snark F_5 with a DRDF with weight 15.

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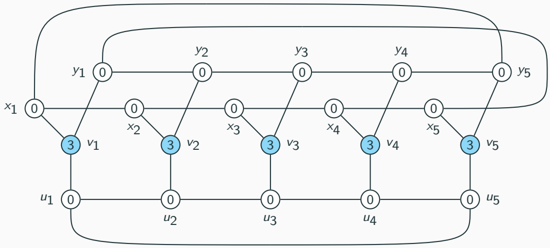

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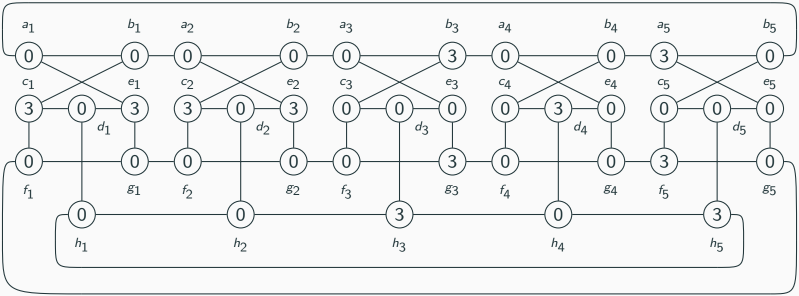
Theorem 5

If F_i is a flower snark, with $i \geq 3$ and i odd, then $\gamma_{dR}(F_i) = 3i$.

Goldberg Snarks

The infinity family of Goldberg Snarks is formed by the graphs $G_3, G_5, G_7, \ldots, G_i$ with $i \geq 3$ and i odd.



Snark G_5 with a DRDF ψ_5 with weight $\omega(\psi_5) = 33$.

Goldberg Snarks — Result

Theorem 6

Let G_i be a Goldberg snark. Then

$$\gamma_{dR}(G_i) \leq \begin{cases} 20 & \text{if } i = 3; \\ 33 & \text{if } i = 5; \\ \frac{13i+3}{2} & \text{if } i \geq 7. \end{cases}$$

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Let G_i be a Goldberg snark. Then

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Theorem 7

If G_i is a Goldberg snark, with $i \geq 3$ and i odd, then $\gamma_{dR}(G_i) \geq 6i + 2$.

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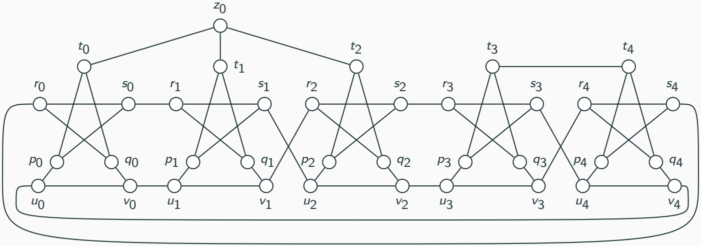
If G_i is a Goldberg snark, with $i \geq 3$ and i odd, then $\gamma_{dR}(G_i) \geq 6i + 2$.

- This lower bound is tight for G_3. That is, $\gamma_{dR}(G_3) = 20$.

Loupekine Snarks

The infinity family of Loupekine snarks is formed by the graphs $GL_3, GL_5, \ldots, GL_i$ with i odd and $i \geq 3$.



A Loupekine snark GL_3 with 5 basic blocks.

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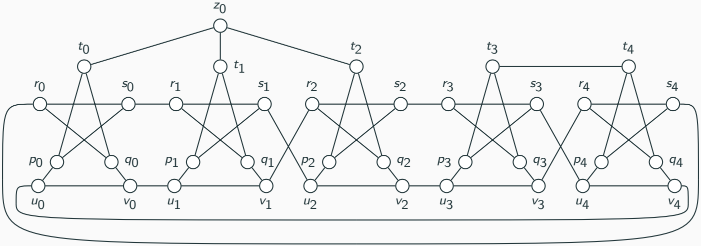

A Loupekine snark GL_3 with 5 basic blocks.

Theorem 8

If GL_i is a Loupekine snark with odd $i \geq 3$ and n vertices, then $\lceil \frac{3n}{4} \rceil + 1 \leq \gamma_{dR}(GL_i) \leq 6i$.

Concluding Remarks

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Open Problems:

1. Conjecture: $\gamma_{dR}(G_i) = \frac{13i+3}{2}$ for all Goldberg snark with $i \geq 7$.

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2. Study the parameter γ_{dR} for other families of cubic graphs.

Thank you

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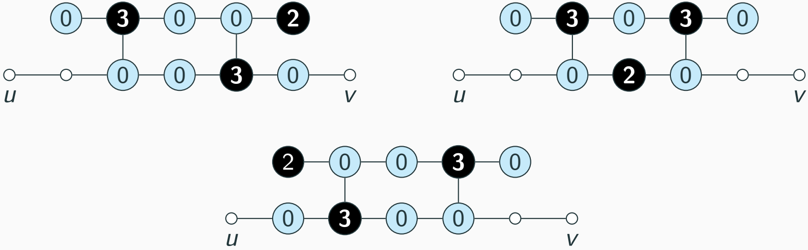
NP-Hardness — Lower Bound

By case analysis, we prove that any DRDF with weight $\gamma_{dR}(H)$ must assign one of the following partial labelings to each gadget of H:



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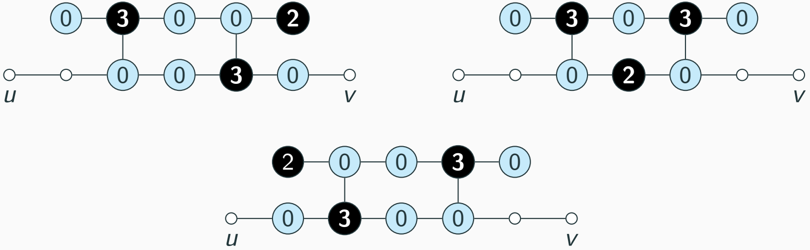

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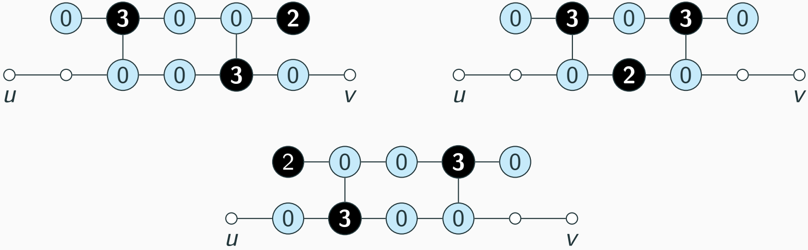

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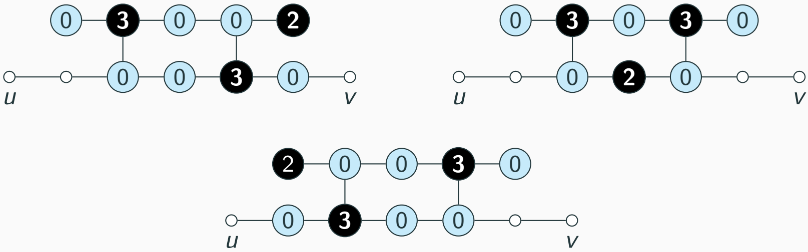

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- Therefore, $\gamma_{dR}(H) = \tau(G) + 2|V(G)| + 8|E(G)|$. ■